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cision was made, it was unalterable. He never made an open profession of religion, yet he was an intensely religious man. He rested his hopes on the sacrifice of the tender and loving Saviour, and we feel satisfied that he has entered into that rest which remaineth for the people of God.

Professor Seitz was not only a mathematician, but he was eminently proficient in other branches of knowledge. His mind was cast in a gigantic mold, "Being devout in heart as well as great in intellect, 'signs and quantities were to him but symbols of God's eternal truth' and he 'looked up through nature up to nature's God.'" Professor Seitz, in the very appropriate words of Dr. Peabody, regarding Benjamin Peirce, Professor of Mathematics and Astronomy in Harvard University, 'saw things precisely as they are seen by the infinite mind. He held the scales and compasses with which the eternal wisdom built the earth and meted out the heavens. As a mathematician, he was adored with awe. As a man, he was a christian in the whole aim and tenor of life.' " No mathematician was so universally loved and honored by his contemporaries as was Professor Seitz.

He did not gain his knowledge from books, for his library consisted of only a few books and periodicals. He gained such a profound insight into the subtle relation of numbers by close application with which he was particularly gifted. He was not a mathematical genius, that is, as usually understood, one who is born with mathematical powers fully developed; but he was a genius in that he was especially gifted with the power to concentrate his mind upon any subject he wished to investigate. This happy faculty of concentrating all his powers of mind upon one topic to the exclusion of all others and viewing that topic from all sides, enabled him to proceed with certainty. Thread by thread and step by step, he took up, and followed out, long lines of thought and arrived at correct conclusions. The darker and more subtle the question appeared to the average mind, the more eagerly he investigated it. No conditions were so complicated as to discourage him.

He left a wife and four sons, one of whom has gone to join his father in the realms of eternal peace. His mother, now (1894) eighty-six years old is still living and enjoying good health.

LOWEST INTEGERS REPRESENTING SIDES OF A RIGHT TRIANGLE.

By LEONARD E. DICKSON, B. Sc., Fellow in Pure Mathematics, University of Texas.

Let the whole numbers expressing the lengths of the sides of a right-angled triangle be reduced to their lowest forms by dividing out their highest common divisor.

Call the resulting numbers a , b , and c .

1. They can not all be *even* numbers. For if so, they would still have the common divisor 2.

2. They can not all be *odd* numbers. For $a^2 + b^2 = c^2$; and, if a and b are odd, their squares are odd, and the sum of their squares even. But c^2 being even, c must be even.

3. c , the greatest of the three, must be *odd*; and, of the remaining two, one must be even and the other odd.

$b^2 = c^2 - a^2 = (c+a)(c-a)$. Suppose b is even. The product of $(c+a)$ and $(c-a)$ is even; hence, both factors are even. [For the case in which one factor is even and the other odd is impossible, since the sum and the difference of two numbers are both even or both odd.] Hence, since the sum and difference of a and c are both even, they are themselves both even or both odd. But by 1, a and c can not both be even if b is even. Hence, when b is even, a and c are both odd.

Similarly, $a^2 = c^2 - b^2 = (c-b)(c+b)$. Reasoning exactly as before, we find that when a is even, b and c are both odd. Hence, we may conclude that c is always odd; and that b is even when a is odd, but odd when a is even.

4. As a corollary to 3, we get the theorem: When the sides of a right triangle are expressed by integers, whether reduced by dividing out their common divisor or not so reduced, the perimeter of the triangle is expressed by an *even* number.

Formulae for obtaining values for a , b , and c .

5. *Rule of Pythagoras:* n , $\frac{n^2-1}{2}$, and $\frac{n^2+1}{2}$, when n is *odd*. The number $\frac{n^2-1}{2}$ is always divisible by 4. For, writing $(2k+1)$ for the odd number n , we get $\frac{n^2-1}{2} = \frac{4k^2+4k}{2} = 2k(k+1)$, where $k(k+1)$ is even whether k be even or odd. As a corollary, the area of the triangle formed is expressed by an *even* number.

6. *Plato's Rule:* m , $\frac{m^2}{4}-1$, $\frac{m^2}{4}+1$, where m is *even*. THEOREM: m is always divisible by 4. For $\frac{m^2}{4}+1$ must [by 3.] be odd. Thus $\frac{m^2}{4}$ is even. Hence, m^2 , being divisible by 8, must be divisible by 16, in order to be a perfect square. As before, the area of the triangle formed is expressed by an *even* number.

7. *Euclid's Rule:* $\sqrt{xy}$, $\frac{x-y}{2}$, $\frac{x+y}{2}$, where x and y are both even or both odd, and where xy is a perfect square. Further, x and y must have no common factor greater than 2; for if so, the sides a , b , and c would contain this factor.

8. *Rule of Maseres:* m , $\frac{m^2-n^2}{2n}$, $\frac{m^2+n^2}{2n}$. Since the last number is an integer, (m^2+n^2) is even and, hence, m and n are both even or both odd. m being one number, n is evidently the difference of the other two. [The numbers found by Maseres' Rule are usually expressed as (m^2-n^2) , $2mn$, and (m^2+n^2) , derived from those above by multiplying through by $2n$. As they thus contain a common factor, they are replaced by the ones given above, which are in their lowest form.]

Correlation of the Four Rules.

9. The Rule of Pythagoras and Plato's Rule are only special cases of the Rule of Maseres; while Euclid's Rule is that of Maseres under a different form. For let $n=1$ in the numbers found by the Rule of Maseres, viz.; m , $\frac{m^2-n^2}{2n}$, and $\frac{m^2+n^2}{2n}$, where m and n are both even or both odd, and we get the numbers given by the Rule of Pythagoras, viz.; m , $\frac{m^2-1}{2}$, $\frac{m^2+1}{2}$, where m is *odd*. Let $n=2$, and we get the numbers given by Plato's Rule; m , $\frac{m^2}{4}-1$, and $\frac{m^2}{4}+1$, where m is *even*. Let $n=y$ and $\frac{m^2}{n}=x$, and we get the numbers given by Euclid's Rule; $\sqrt{xy}$, $\frac{x-y}{2}$, $\frac{x+y}{2}$, where x and y are both even or both odd, and where xy is a perfect square (m^2).

A New Rule.

10. Let a , b , and c be the numbers reduced to their lowest form which represent the sides of a right-angled triangle. Let $(c-a)=m$, $(c-b)=n$, where m and n are integers, one being even and the other odd.

$$\begin{aligned}\therefore (c-m)^2 + (c-n)^2 &= c^2, \\ c^2 - 2c(m+n) + m^2 + n^2 &= c^2, \\ c^2 - 2c(m+n) + (m+n)^2 &= 2mn, \\ c - (m+n) &= \pm \sqrt{2mn}, \\ c &= m+n \pm \sqrt{2mn}.\end{aligned}$$

Since $2c=(a+b)+(m+n)$ and $c < (a+b)$; $\therefore c > (m+n)$. Hence, we must take the plus sign before the radical.

$$\begin{aligned}\therefore c &= m+n+\sqrt{2mn}; \\ a &= m+\sqrt{2mn}; \text{ and} \\ b &= n+\sqrt{2mn}.\end{aligned}$$

By inspection, m and n must have no common factor; for this factor would appear in a , b , and c . My rule may be stated formally, thus:

Take any two integers, m and n , such that one is even and the other odd, and such that their product is a perfect square; then the three sides will be $m+\sqrt{2mn}$, $n+\sqrt{2mn}$, $m+n+\sqrt{2mn}$.

By Euclid's Rule (to which the other three are special cases or equivalent), we take any two integers, m and n , such that both are even or both odd, and such that their product is a perfect square; then the sides are $\sqrt{mn}$, $\frac{m-n}{2}$, $\frac{m+n}{2}$.

Complete Table of Values.

11. We may get an absolutely complete table of values for a , b , and c , and one in which each set is reduced to its lowest form, by employing either Euclid's Rule or my rule, in each case observing the necessary conditions and restrictions.

In Euclid's set of values, $\sqrt{xy}$, $\frac{x-y}{2}$, and $\frac{x+y}{2}$, x and y must both be even or both odd; and further they can have no common divisor higher than 2, since a , b , and c would then have this divisor. The *practical* rule, which gives every possible set of values in their lowest form, and which gives each set but once is as follows: Let y be any *odd* square number whatever, and x any greater *odd* square number not divisible by y . Thus if $y=9$, x may be 25, 49, 121, 169, etc.

In my set of values, $m + \sqrt{2mn}$, $n + \sqrt{2mn}$, $m+n + \sqrt{2mn}$, m and n must one be even and the other odd; and further, they can have no common divisor whatever, since a , b , and c would then have this divisor. My *practical* rule, which gives every possible set of values in their lowest form, and which gives this set but once, is this: Let m be any *odd* square number whatever, and let n be the *double* of any square number whatever not divisible by m . Thus if $m=9$, n may be the *double* of 1, 4, 16, 25, 49, 64, 100, etc.

Additional Theorems.

12. One out of each set of values found by my practical rule is divisible by 4. Let $m=(2l+1)^2$ and $n=2k^2$. Then $\sqrt{2mn}=2k(2l+1)$. Thus $n + \sqrt{2mn} = 2k^2 + 2k(2l+1) = 2k(2l+2) = 4k(l+1)$.

13. One out of each set of numbers found by the practical rule derived from Euclid's Rule is divisible by 4. Let $y=(2l+1)^2$ and $x=(2k+2)^2$. $\frac{x-y}{2} = 2(l^2 - k^2 + l - k) = 2(l-k)(l+k+1)$. Now since $(l-k)$ and $(l+k)$ are both even or both odd, $(l-k)$ and $(l+k+1)$ are one even and the other odd. $\therefore \frac{x-y}{2}$ is divisible by 4.

14. As a corollary, the area of any right-angled triangle whose sides are integers is express by an *even* integer.

15. The radii of the inscribed and three escribed circles of any right-triangle whose sides are integers are all expressed by integers. The trigonometric expressions for r , r_a , r_b , r_c , viz.; $\frac{\Delta}{s}$, $\frac{\Delta}{s-a}$, $\frac{\Delta}{s-b}$, $\frac{\Delta}{s-c}$, become for the right-triangle $(s-c)$, $(s-b)$, $(s-a)$, and s respectively, where s is the semi-perimeter (always an integer). Hence, the radii are all integers.

Dr. Halsted's *Mensuration* contains a three-page table of values for a , b , and c . It is absolutely *complete* as far as the 59th set; but beyond this, sets are occasionally omitted.

SCALENE TRIANGLES.

Let a , b , and c be three integers having no common divisor which represent the sides of a scalene triangle whose *area* may be expressed by an *integer*.

1. They can not all be even.
2. They can not all be odd.

For $\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{4} \sqrt{2(a^2b^2 + a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4)}$. Now

if a , b , and c all be odd, $(a^4 + b^4 + c^4)$ is odd and the expression under the radical sign must be odd. Thus the area could not be expressed by an integer.

3. No two of them can be even. Proved as in (2).

4. Hence, of the three, one is even and the other two odd.

5. As a corollary, the perimeter of the triangle will be expressed by an *even* number.

6. THEOREM: The area of the triangle is expressed by an *even* number. Let $a=2l+1$; $b=2m+1$; $c=2n$, as one of the sides is even and the other two odd.

$\therefore \frac{a+b+c}{2} = l+m+n+1$; $\frac{a+b-c}{2} = l+m-n+1$; $\frac{a-b+c}{2} = l-m+n$; $-\frac{a+b+c}{2} = -l+m+n$. Hence, of these four quantities, two are even and two odd.

$\therefore$ area, $= \sqrt{\frac{a+b+c}{2} \cdot \frac{a+b-c}{2} \cdot \frac{a-b+c}{2} \cdot \frac{-a+b+c}{2}}$, is divisible by 2.

7. Problem: To find three integers a , b , and c representing the sides of a triangle whose *area* may be expressed by an *integer*.

$$\Delta = \frac{1}{4} \sqrt{2(a^2b^2 + a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4)}.$$

Solving for a^2 , we get $a^2 = b^2 + c^2 \pm 2\sqrt{b^2c^2 - 4\Delta^2}$.

Similarly, $b^2 = a^2 + c^2 \pm 2\sqrt{a^2c^2 - 4\Delta^2}$.

Similarly, $c^2 = a^2 + b^2 \pm 2\sqrt{a^2b^2 - 4\Delta^2}$.

Let a be greater than either b or c . Then $\angle A$ may be acute or obtuse; while $\angle B$ and $\angle C$ are both acute. Hence, writing l for $-\sqrt{b^2c^2 - 4\Delta^2}$, m for $-\sqrt{a^2c^2 - 4\Delta^2}$, n for $-\sqrt{a^2b^2 - 4\Delta^2}$, l may be $\pm$, while m and n must be $+$. Further, the conditions of the problem require that l , m , and n be integers. We have $a^2 - b^2 - c^2 = \mp 2l$; $b^2 - a^2 - c^2 = -2m$; $c^2 - a^2 - b^2 = -2n$. $\therefore a^2 = m + n$; $b^2 = n \pm l$; $c^2 = m \pm l$. Thus the conditions that $b^2c^2 - 4\Delta^2$, $a^2c^2 - 4\Delta^2$, $a^2b^2 - 4\Delta^2$, be perfect squares, l^2 , m^2 , and n^2 , respectively, reduce to the single condition that $4\Delta^2 = mn \pm ml \pm nl$. Hence, the Rule to find a , b , c such that area $\triangle ABC$ be integral is: Take any three integers $\pm l$, m , and n , such that one is odd and the other two are even, and such that the sums of them two at a time are perfect squares, and lastly such that the sum of their products two at a time is a perfect square; then $a = \sqrt{m+n}$; $b = \sqrt{m \pm l}$; $c = \sqrt{m \pm l}$; and $\Delta = \frac{1}{2} \sqrt{mn \pm ml \pm nl}$.

8. Let fall a $\perp$ from each vertex of the triangle upon the opposite side. We can express the intercepts on the sides either in terms of l, m , and n or of a, b , and c . For example, those on c are $\frac{\pm l}{\sqrt{m \pm l}}$ and $\frac{m}{\sqrt{m \pm l}}$ or in terms of a, b, c , $\frac{b^2 + c^2 - a^2}{2c}$ and $\frac{a^2 + c^2 - b^2}{2c}$. Thus in general these intercepts are not integers.

9. If two right-triangles, having a perpendicular side of the one equal to a perpendicular side of the other, be placed so that these sides coincide, the other perpendicular sides falling in the same straight, then one of the two possible scalene triangles are formed. The sides and the area of these scalene tri-

angles are expressed by integers, providing the three sides of each triangle be integers. But from (8) we find that the sides of the new triangles may have a common factor. Guarding this point, we may form from a table of the lowest integers representing sides of a right triangle a corresponding table for a scalene triangle whose *area is an integer*. A four-page table of this kind is given in Dr. Halsted's *Mensuration*.



POSTULATE I. OF EUCLID'S ELEMENTS.

By Professor JOHN L. LYLE, Ph. D., Westminister College, Fulton, Missouri.



"Let it be granted that a straight line may be drawn from any one point to any other point." Euclid lays down the statement just quoted as his first postulate regulative of geometrical constructions. Wherever any two points may be located in unbounded space, Euclid assumes that a straight line may be drawn from one of them to the other.

Such a line is finite in length, of course, according to the definition that "a *finite* straight line is one that has *two ends*".

The assumption that a straight line of *infinite* (boundless) length can be drawn between two points in space is not only anti-Euclidean but also destructive of the logical law of non-contradiction which forbids that contradictory marks shall be attributed to any straight line whatever. A line can not have two ends and at the same time be without ends, (infinite, that is, unbounded).

In every rectilineal triangle, there are three angular points and each of the three sides may be constructed in strict harmony with the 1st postulate of Euclid. Each of these sides has the distinctive marks of a finite straight line, to wit: two ends.

Euclid proves in proposition XVII, Book I., that "Any two angles of a triangle are together less than two right angles". This proposition is the converse of Euclid's 12th axiom about which so much has been written.

John Playfair states that axiom as follows: "If a straight line meet two straight lines, so as to make the interior angles on the same side of it less than two right angles, these straight lines being continually produced will at length meet on the side on which the angles are less than two right angles."

Lobatschewsky, in his theorem 19, demonstrates the proposition that the angle-sum of a rectilineal triangle can not be greater than two right angles. Then assuming the falsity of Euclid's 12th axiom, he concludes (whether logically or illogically) that the angle-sum of a rectilineal triangle is less than two right angles.